Turbulent Flows
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Solution to Exercise 2.2

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The velocity of a particle is defined as

$$\mathbf{U}^+(t, \mathbf{Y}) \equiv \frac{d\mathbf{X}^+(t, \mathbf{Y})}{dt}, \tag{1}$$

and the velocity field $\mathbf{U}(\mathbf{x}, t)$ is defined by the velocities of particles at all postions $\mathbf{x} = \mathbf{X}^+$ for all t, so that

$$\mathbf{U}^+(t, \mathbf{Y}) = \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y}), t). \tag{2}$$

Thus,

$$\begin{aligned} \frac{d\mathbf{s}}{dt} &= \frac{d\mathbf{X}^+(t, \mathbf{Y} + d\mathbf{Y})}{dt} - \frac{d\mathbf{X}^+(t, \mathbf{Y})}{dt} \\ &= \mathbf{U}^+(t, \mathbf{Y} + d\mathbf{Y}) - \mathbf{U}^+(t, \mathbf{Y}) \\ &= \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y} + d\mathbf{Y}), t) - \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y}), t). \end{aligned} \tag{3}$$

Invoking the definition of $\mathbf{s}(t)$, we write

$$\mathbf{X}^+(t, \mathbf{Y} + d\mathbf{Y}) = \mathbf{X}^+(t, \mathbf{Y}) + \mathbf{s}(t), \tag{4}$$

and expand the first term of Eq.(3) as a Taylor series as follows:

$$\begin{aligned} \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y} + d\mathbf{Y}), t) &= \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y}) + \mathbf{s}(t), t) \\ &= \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y}), t) + \mathbf{s} \cdot (\nabla \mathbf{U})_{\mathbf{x}=\mathbf{X}^+(t,\mathbf{Y})} \\ &+ \mathcal{O}(s^2). \end{aligned} \tag{5}$$

Neglecting higher order terms, we substitute Eq.(5) into Eq.(3) and find

$$\begin{aligned} \frac{d\mathbf{s}}{dt} &= \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y}), t) + \mathbf{s} \cdot (\nabla \mathbf{U})_{\mathbf{x}=\mathbf{X}^+(t,\mathbf{Y})} - \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y}), t) \\ &= \mathbf{s} \cdot (\nabla \mathbf{U})_{\mathbf{x}=\mathbf{X}^+(t,\mathbf{Y})}. \end{aligned} \tag{6}$$