

**Turbulent Flows**  
 Stephen B. Pope  
*Cambridge University Press* (2000)

**Solution to Exercise 2.2**

*Prepared by:* Allie King

*Date:* 3/7/03

The velocity of a particle is defined as

$$\mathbf{U}^+(t, \mathbf{Y}) \equiv \frac{d\mathbf{X}^+(t, \mathbf{Y})}{dt}, \quad (1)$$

and the velocity field  $\mathbf{U}(\mathbf{x}, t)$  is defined by the velocities of particles at all positions  $\mathbf{x} = \mathbf{X}^+$  for all  $t$ , so that

$$\mathbf{U}^+(t, \mathbf{Y}) = \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y}), t). \quad (2)$$

Thus,

$$\begin{aligned} \frac{d\mathbf{s}}{dt} &= \frac{d\mathbf{X}^+(t, \mathbf{Y} + d\mathbf{Y})}{dt} - \frac{d\mathbf{X}^+(t, \mathbf{Y})}{dt} \\ &= \mathbf{U}^+(t, \mathbf{Y} + d\mathbf{Y}) - \mathbf{U}^+(t, \mathbf{Y}) \\ &= \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y} + d\mathbf{Y}), t) - \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y}), t). \end{aligned} \quad (3)$$

Invoking the definition of  $\mathbf{s}(t)$ , we write

$$\mathbf{X}^+(t, \mathbf{Y} + d\mathbf{Y}) = \mathbf{X}^+(t, \mathbf{Y}) + \mathbf{s}(t), \quad (4)$$

and expand the first term of Eq.(3) as a Taylor series as follows:

$$\begin{aligned} \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y} + d\mathbf{Y}), t) &= \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y}) + \mathbf{s}(t), t) \\ &= \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y}), t) + \mathbf{s} \cdot (\nabla \mathbf{U})_{\mathbf{x}=\mathbf{X}^+(t, \mathbf{Y})} \\ &\quad + \mathcal{O}(s^2). \end{aligned} \quad (5)$$

Neglecting higher order terms, we substitute Eq.(5) into Eq.(3) and find

$$\begin{aligned} \frac{d\mathbf{s}}{dt} &= \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y}), t) + \mathbf{s} \cdot (\nabla \mathbf{U})_{\mathbf{x}=\mathbf{X}^+(t, \mathbf{Y})} - \mathbf{U}(\mathbf{X}^+(t, \mathbf{Y}), t) \\ &= \mathbf{s} \cdot (\nabla \mathbf{U})_{\mathbf{x}=\mathbf{X}^+(t, \mathbf{Y})}. \end{aligned} \quad (6)$$

This work is licensed under the Creative Commons Attribution-NonCommercial-ShareAlike License. To view a copy of this license, visit <http://creativecommons.org/licenses/by-nc-sa/1.0> or send a letter to Creative Commons, 559 Nathan Abbott Way, Stanford, California 94305, USA.